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THE NATURAL VIBRATIONS OF A MOMENTLESS SPHERICAL SHELL†

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The problem of the natural vibrations of a momentless spherical shell is investigated. It is reduced to the solution of a system of ordinary differential equations of the fourth order in the variables. It is shown that there are two classes of solution of this system: (1) all three components of the displacements are non-zero, and (2) only the tangential components are non-zero. For each of these classes the system of equations is reduced to solving second-order equations, i.e. in each class there are two linearly independent solutions. A regular solution is obtained for class 1, expressed in associated Legendre functions, and a solvable equation for class 2. The two classes of solution obtained enable one to solve the problem of the vibrations of a momentless spherical shell fairly effectively for any tangential boundary conditions.

THE EQUATIONS of motion of a momentless spherical shell and the boundary conditions in standard notation can be written in the following form

$$\begin{gathered}
\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\left(\frac{\partial u\sin\theta}{\partial\theta}+\frac{\partial v}{\partial\varphi}\right)-(1+\sigma)\frac{\partial w}{\partial\theta}+ \\
+(1-\sigma)\left[u+\frac{1}{2}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\frac{1}{\sin\theta}\left(\frac{\partial u}{\partial\varphi}-\frac{\partial v\sin\theta}{\partial\theta}\right)\right]-\frac{(1-\sigma^2)\rho R^2}{E}\frac{\partial^2 u}{\partial t^2}=0 \\
\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\frac{1}{\sin\theta}\left(\frac{\partial u\sin\theta}{\partial\theta}+\frac{\partial v}{\partial\varphi}\right)-\frac{(1+\sigma)}{\sin\theta}\frac{\partial w}{\partial\varphi}- \\
-(1-\sigma)\left[v+\frac{1}{2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\left(\frac{\partial v\sin\theta}{\partial\theta}-\frac{\partial u}{\partial\varphi}\right)\right]-\frac{(1-\sigma^2)\rho R^2}{E}\frac{\partial^2 v}{\partial t^2}=0 \\
\frac{1+\sigma}{\sin\theta}\left(\frac{\partial u\sin\theta}{\partial\theta}+\frac{\partial v}{\partial\varphi}\right)-2(1+\sigma)w-\frac{(1-\sigma^2)\rho R^2}{E}\frac{\partial^2 w}{\partial t^2}=0
\end{gathered} \tag{1}$$

$$\begin{gathered}
k_1N_1=(1-k_1)Eu,\quad k_2S=(1-k_2)Ev\quad\text{for}\quad\theta=\theta_0 \\
0<k_1<1,\quad 0<k_2<1
\end{gathered} \tag{2}$$

where N_1 is the normal meridional force, S is the shear force, and k_1 and k_2 are elastic clamping coefficients.

We will write the relations of elasticity and the expressions for the strains as follows:

$$N_1=\frac{2Eh}{1-\sigma^2}(\epsilon_1+\sigma\epsilon_2),\quad S=\frac{Eh}{1+\sigma}\omega \tag{3}$$

$$\begin{gathered}
\epsilon_1=\frac{1}{R}\left(\frac{\partial u}{\partial\theta}-w\right),\quad \epsilon_2=\frac{1}{R}\left(u\operatorname{ctg}\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\theta}-w\right) \\
\omega=\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{v}{\sin\theta}\right)\right]
\end{gathered} \tag{4}$$

Using the method of separation of variables, we obtain a regular solution of problem (1), (2) at the vertex of the shell for the above-mentioned class 1 in the following form

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$$u = (1+\sigma)C_{mn}(dP_m^n(\theta)/d\theta)\cos n\varphi\cos\omega t$$
$$v = -(1+\sigma)C_{mn}(P_m^n(\theta)/\sin\theta)\sin n\varphi\cos\omega t$$
$$w = [1-\sigma-m(m+1)+(1-\sigma^2)\lambda_n^2]\,C_{mn}P_m^n(\theta)\cos n\varphi\cos\omega t,\quad \lambda_n^2 = \rho R^2\omega^2/E \tag{5}$$

where $P_m^n(\theta)$ are the associated Legendre functions ($n = 1, 2, \ldots$), and m are the roots of the algebraic equation

$$(1-\lambda^2)m(m+1) - [2+(1+3\sigma)\lambda^2 - (1-\sigma^2)\lambda^n] = 0 \tag{6}$$

The problem is solved individually for each number n of waves along parallels of the shell.

Thus, to determine the index m in the associated Legendre functions, in terms of which the solution (5) is expressed, we obtain the quadratic equation (6). Its roots m_1 and m_2 are real, and their sum is equal to -1. We know that in this case the Legendre functions $P_{m_1}^n$ and $P_{m_2}^n$ are linearly dependent. Hence, two linearly independent solutions expressed in terms of associated Legendre functions do not exist.

We will show that in this class of solution (all three components of the displacement are non-zero) a second regular solution at the vertex of the cell does not, in general, exist.

Separating the variables, we will take the solutions of system (1) in the form

$$u = U(\theta)\cos n\varphi\cos\omega t,\quad v = V(\theta)\sin n\varphi\cos\omega t,\quad w = W(\theta)\cos n\varphi\cos\omega t \tag{7}$$

After substituting into system (1) and cancelling factors which depend on φ and t from the third equation, we obtain the expression

$$L(U, V) = (2 - X)W \tag{8}$$
$$L(U, V) = \frac{1}{\sin\theta}\left(\frac{dU\sin\theta}{d\theta} + nV\right),\quad X_\pm = \frac{1-\sigma}{E}\rho R^2\omega^2$$

Substituting this into the first two equations we can write them in the form

$$a\frac{dW}{d\theta} + bU - \frac{n(1-\sigma)}{\sin\theta}L(V, U) = 0 \tag{9}$$
$$-\frac{n}{\sin\theta}aW + bV + (1-\sigma)\frac{dL(V, U)}{d\theta} = 0$$
$$a = 1-\sigma-X_-,\quad b = 1-\sigma+X_-$$

Multiplying the first equation of (9) by $\sin\theta$, differentiating it with respect to θ, and then adding the equation obtained to the second equation of (9), multiplying it first by n, and using Eq. (8), first multiplied by $\sin\theta$, we obtain

$$a\sin\theta\frac{d^2W}{d\theta^2} + a\cos\theta\frac{dW}{d\theta} - \left[\frac{na}{\sin\theta} - b\sin\theta(2-X_-)\right]W = 0 \tag{10}$$

Hence, the resolvent equation has turned out to be a second-order equation, and not a fourth-order equation, as might have been supposed. The regular solution of this equation will be identical with the component W in (5) if we omit the factor $C_{mn}\cos n_\varphi\cos\omega t$ in the expression for it. The second solution of this equation, due to the presence of a singularity for the second derivative in (10) at the vertex of the shell ($\theta = 0$), will be irregular.

For $n \neq 0$, when it is required to satisfy the two boundary conditions (2), it is necessary to have one other regular solution. Since the boundary conditions (2) do not contain the bending function W, we will assume that when $n \neq 0$ there is a second class of solution of system (1) of the form (7) when $W \equiv 0$.

In this connection it is necessary to show that when $W = 0$ the system of three equations for the two unknown functions U and V are compatible and lead to a second-order resolvent equation.

In fact, when $W \equiv 0$, the system takes the form

$$bU - \frac{2n(1-\sigma)}{\sin\theta}L(V, U) = 0$$
$$bV + 2(1-\sigma)dL(V, U)/d\theta = 0,\quad L(U, V) = 0 \tag{11}$$

Multiplying the first equation of (11) by $\sin\theta$ and differentiating it with respect to θ, we can show that the second equation of (11) is a consequence of the first and third.

Now consider the system of first and third equations (11). By finding V from the third equation and substituting it into the first, we obtain a second-order equation for the component U

$$\frac{d^2U}{d\theta^2} + 3\frac{dU}{d\theta}\operatorname{ctg}\theta + (2a - 1 + \operatorname{ctg}^2\theta - \frac{n^2}{\sin^2\theta})\,U = 0$$

Using this equation we can obtain a second regular solution, which will be linearly independent with respect to the previously obtained regular solution (5).

EXAMPLE OF THE CALCULATION

Consider the axisymmetrical vibrations of a shell in the form of a spherical cupola ($n = 0$).

Of the boundary conditions (2) it is only necessary to satisfy the first. For the special case when $u(\theta_0) = 0$, it follows from this equation and the solution (5) that the frequency equation has the form

$$dP^0_m/d\theta = 0 \text{ when } \theta = \theta_0 \tag{12}$$

We will determine the values of the first two frequencies of the axisymmetric free vibrations of a hemispherical cupola with the boundary condition $u(\pi/2) = 0$.

Equation (2) was solved using tables [1]: $m_1 = 1.92$ and $m_2 = 3.95$. Further, using Eq. (6) we obtain $\lambda_{01} = 0.718$ and $\lambda_{02} = 0.920$, which gives values of 1530 and 1960 s^{-1} for the first two frequencies. The free vibrations of a steel hemispherical cupola with a diameter of 76.2 cm and a thickness of 0.159 cm, rigidly clamped along the contour, has been investigated experimentally [2]. The first two frequencies were 1540 and 1880 s^{-1}, which agree well with the calculations.

In conclusion we note that the investigation carried out above enables the frequencies and modes of free momentless vibrations of spherical shells to be determined fairly easily.

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